Discovering Lake Fred: A Journey Through Advanced Calculus

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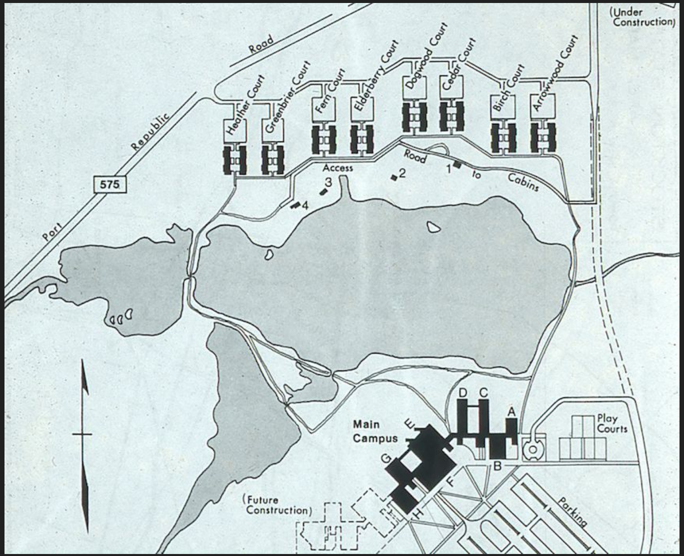
Background on Lake Fred

- **Origins & Early History:**
  - Established between 1860s-1870s
  - Vital for early cranberry production
- **Industrial Use:**
  - Operated by the Sawmill Corporation
  - Transition to office use with log cabins in 1957
- **Academic Chapter:**
  - Stockton State College arrives in 1971
  - Growth to Stockton University, with Lake Fred as a central symbol
- **Lake Today:**
  - Hub for student activities and community events
  - Ecological efforts and wildlife conservation



1972 Map of the Campus

Video of Lake Fred Using DJI Mini 3 Pro (ft. Kenny G.)

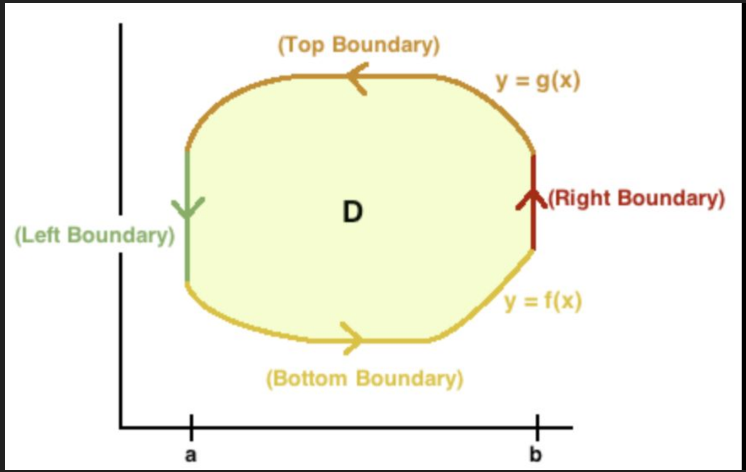
What Is the Premise of Our Research?

- **Objective**: To accurately measure the area of Lake Fred.
- **Approach**: Utilize multivariable calculus principles in real-world applications.
- **Methodology**:
  - Investigate various techniques for calculating Lake Fred's area.
  - Perform comparative analysis on the effectiveness of these methods.
- **Rationale**:
  - To understand and implement the practical use of Green’s Theorem from Calculus III.

Why?

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

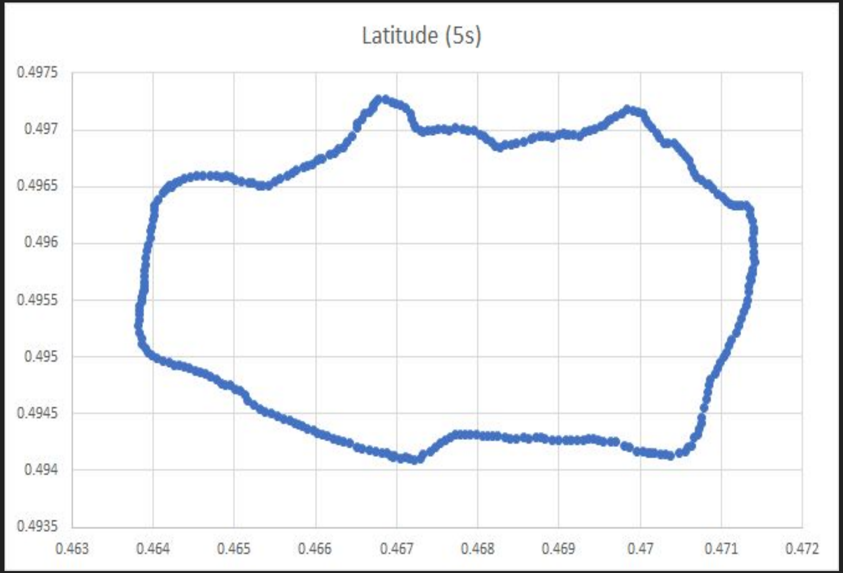
- Learned about Green's Theorem in Calculus III
- Green's Theorem relates a line integral around a simple closed curve to a double integral over a simply connected curve.
- We use Green's Theorem to calculate the line integrals of a closed curve.
- Lake Fred represents a closed curve
- We wanted to apply Green's Theorem to a practical life situation



General Procedure

- **Data Collection**:
    - Obtained GPS coordinates encircling Lake Fred (Vernier LabQuest 3)
    - Translated GPS data to Cartesian points for computational analysis.
- **Area Calculation Techniques**:
    - **Method 1 (Calculus II)**:
        - Halved the data points.
        - Applied curve fitting techniques to approximate the area.
    - **Method 2 (Calculus III)**:
        - Parametrized the lake's perimeter as a simple closed curve.
        - Implemented Green's theorem to find the enclosed area.
    - ***NEW* Method 3 (Numerical Analysis)**

Preliminary Results With the Vernier LabQuest 3



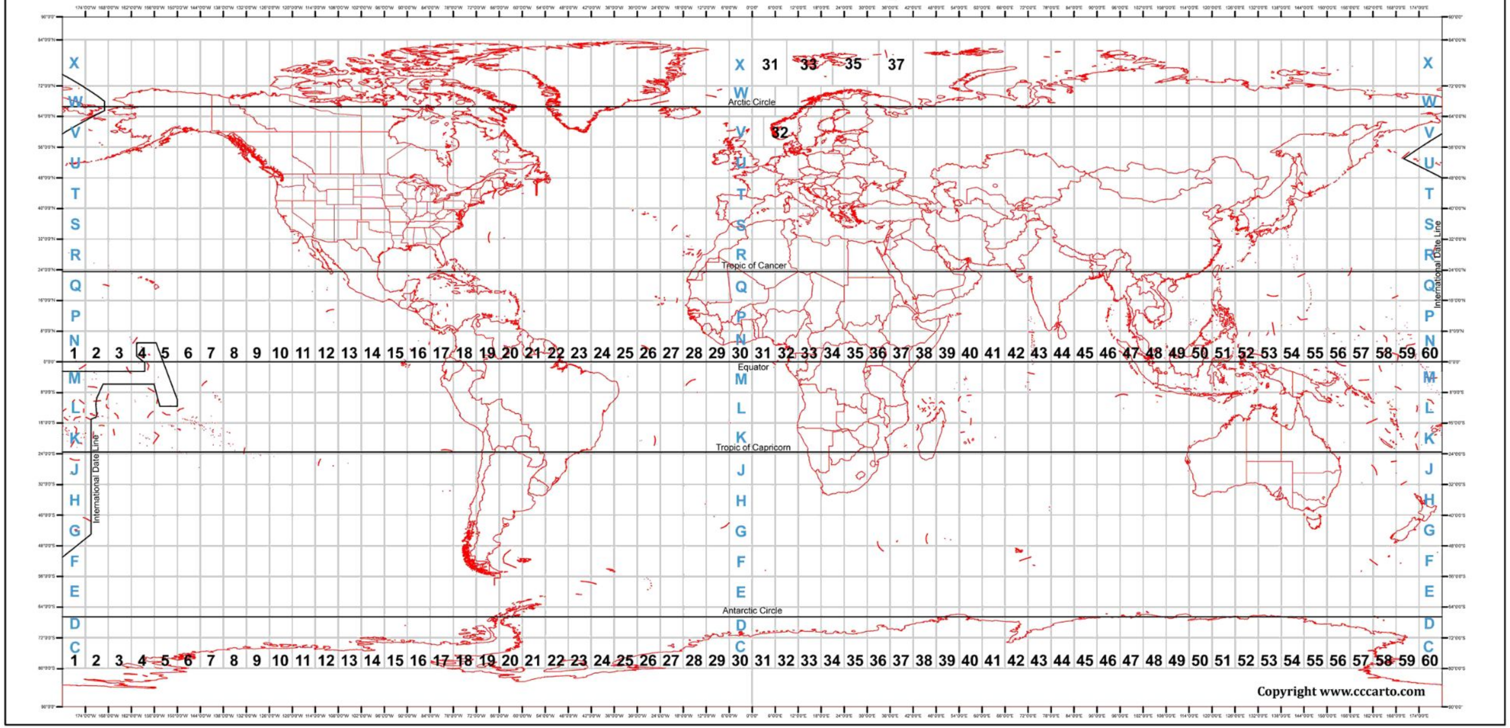
- Found GPS Coordinates for every 1 second.
- Graphed the points for every 5 seconds.

Units of Measurement

Units of Measurement: UTM

- Universal Transverse Mercator is a plane coordinate grid system that consists of 60 zones, each 6-degrees of longitude in width
- Provides a constant distance relationship anywhere on the map
- Coordinates are measured as northings and eastings in meters.
- UTM northing coordinates
  - are measured relative to the equator.
  - South to North are measured from C to X.
- UTM easting coordinates
  - are referenced to the centerline of the zone (central meridian).
  - West to East are measured from 1 to 60

World UTM Zone Map



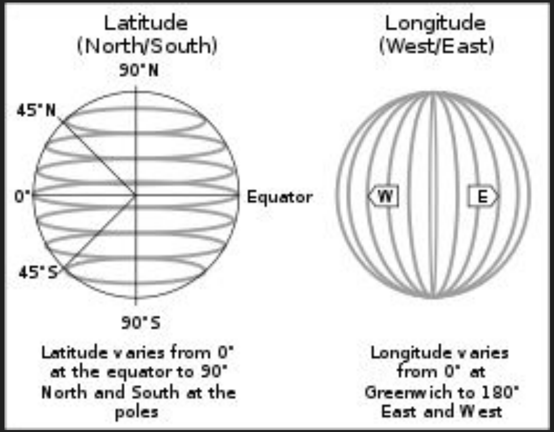
UTM data from the Vernier LabQuest 3

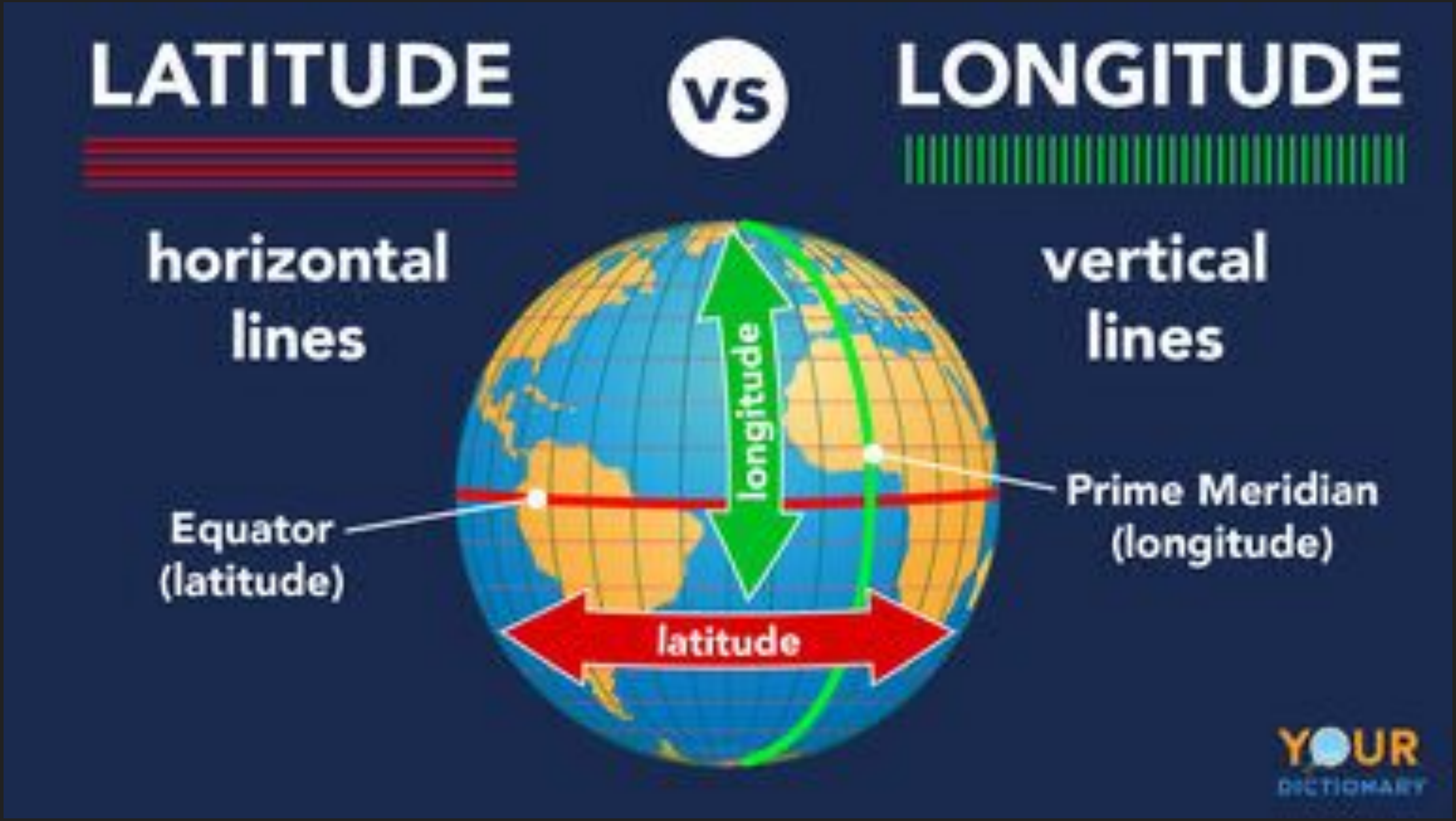
Vernier Format 2
LabQuest Data.txt 7/22/2022 12:54:37
Run 1

Time	Northing	Easting	Altitude	Speed	Direction
T	N	E	A	S	D
s	m	m	m	m/s	°
0	4376081 [S]	552449 [18]	15	0	55
1	4376081 [S]	552450 [18]	14	0	55
2	4376082 [S]	552450 [18]	14	1	22
3	4376083 [S]	552451 [18]	13	1	23
4	4376083 [S]	552451 [18]	13	1	15
5	4376085 [S]	552451 [18]	13	1	358
6	4376086 [S]	552452 [18]	13	1	8
7	4376087 [S]	552453 [18]	13	1	16
8	4376087 [S]	552453 [18]	13	1	3
9	4376088 [S]	552453 [18]	13	1	13
10	4376089 [S]	552454 [18]	12	1	19
11	4376090 [S]	552454 [18]	12	1	9
12	4376091 [S]	552454 [18]	12	1	16
13	4376092 [S]	552454 [18]	12	1	8
14	4376093 [S]	552454 [18]	12	1	3
15	4376094 [S]	552454 [18]	12	1	4
16	4376095 [S]	552454 [18]	13	1	4
17	4376096 [S]	552454 [18]	13	1	8
18	4376097 [S]	552455 [18]	12	1	8
19	4376098 [S]	552455 [18]	12	1	13
20	4376099 [S]	552456 [18]	12	1	12
21	4376100 [S]	552456 [18]	12	1	15
22	4376101 [S]	552456 [18]	12	1	2
23	4376102 [S]	552456 [18]	11	1	2
24	4376103 [S]	552456 [18]	12	1	0
25	4376104 [S]	552455 [18]	12	1	353
26	4376105 [S]	552455 [18]	11	1	354
27	4376107 [S]	552455 [18]	11	1	7
28	4376108 [S]	552455 [18]	12	1	9

Units of Measurement: Longitude and Latitude

- A spherical coordinate system for measuring and communicating positions directly on the Earth.
- Latitude is a measurement on a globe of location north or south of the Equator.
  - Length of a degree of arc of latitude is approximately 111 km.
- Longitude is a measurement of location east or west of the prime meridian that passes through both poles.
  - Measured 180 degrees both east and west of the prime meridian
  - The distance per degree of longitude at the Equator is about 111.32 km and 0 km at the poles.



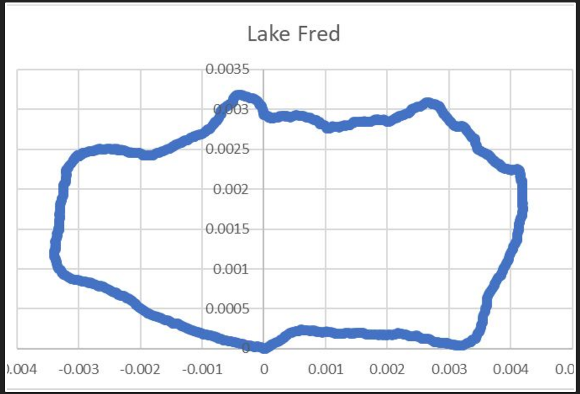

LATITUDE
vs
LONGITUDE
horizontal lines
vertical lines
longitude
Prime Meridian (longitude)
Equator (latitude)
latitude
YOUR DICTIONARY

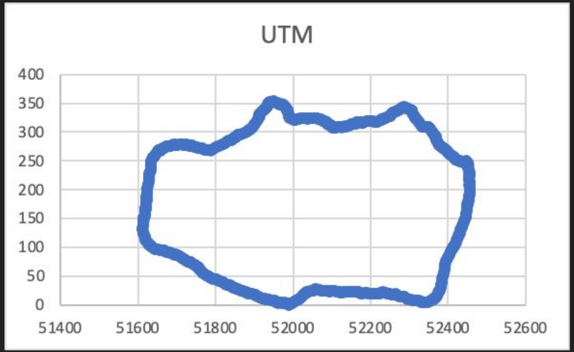
Difficulties With Units of Measurement

- Longitude and Latitude
  - What are our units?
  - How do we translate it to a 2D plane?

Shifting Coordinates

- Longitude and Latitude
  - Found coordinate with minimum y values
  - Shifted all coordinates to make point the new origin
- UTM
  - Subtracted 500,000 from each easting value to make centerline the new origin
  - Found the minimum northing value and shifted points to make that the new origin





Conversion Factor for Longitude and Latitude

- Two points were taken from our data set and were used to find a conversion factor.
- Found the distance (°) between the points using distance formula
- Found the distance (km) between the points w/ a distance calculator (used Vincenty’s algorithm)
- Divided the distance in kilometers by the distance in degrees
- Converted from km/° to m/°
- Final conversion factor : 110.83 km/° or 110831 m/°

Conversion Factor

Input Data

Lat1		Lon1	
39.49566	N	74.52866	W
Lat2		**Lon2**	
39.4963	N	74.5286	W

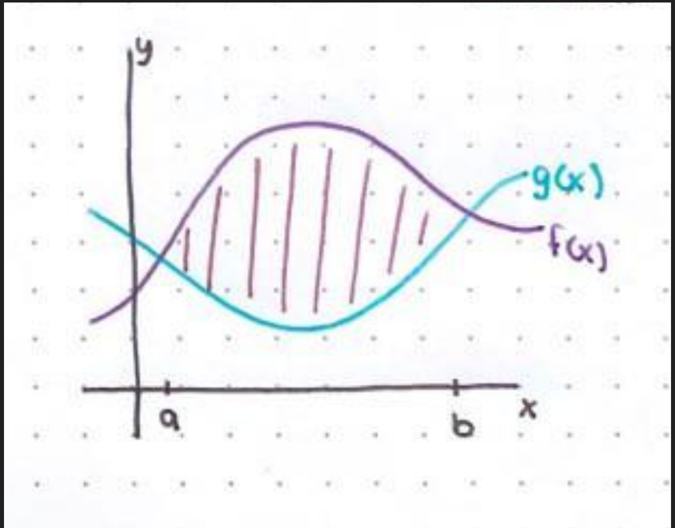
Output

Course 1-2	Course 2-1	Distance
4.154308756	184.15434691	0.0712431647

Calculus II Method

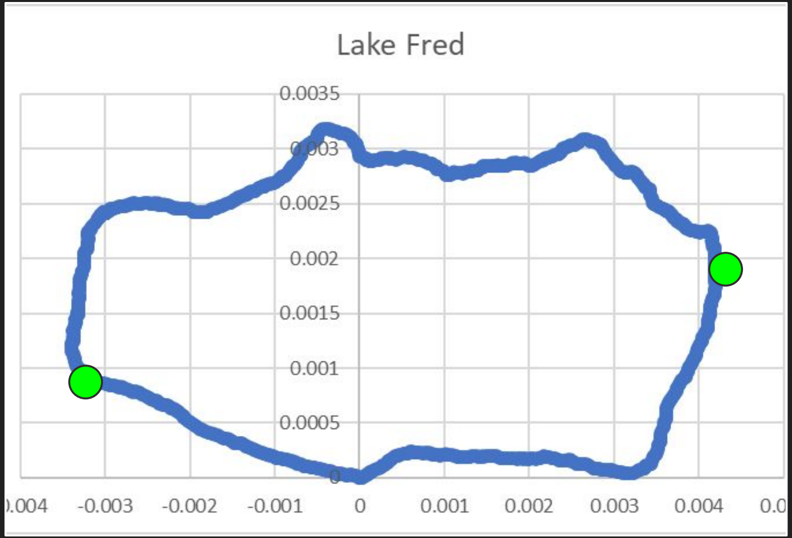
A Brief Review of Integrals

- Integrals find the area under a function
- If we have a region R bounded by f(x) and g(x), where f(x) is the top function:
- $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \text{Enclosed Area}$



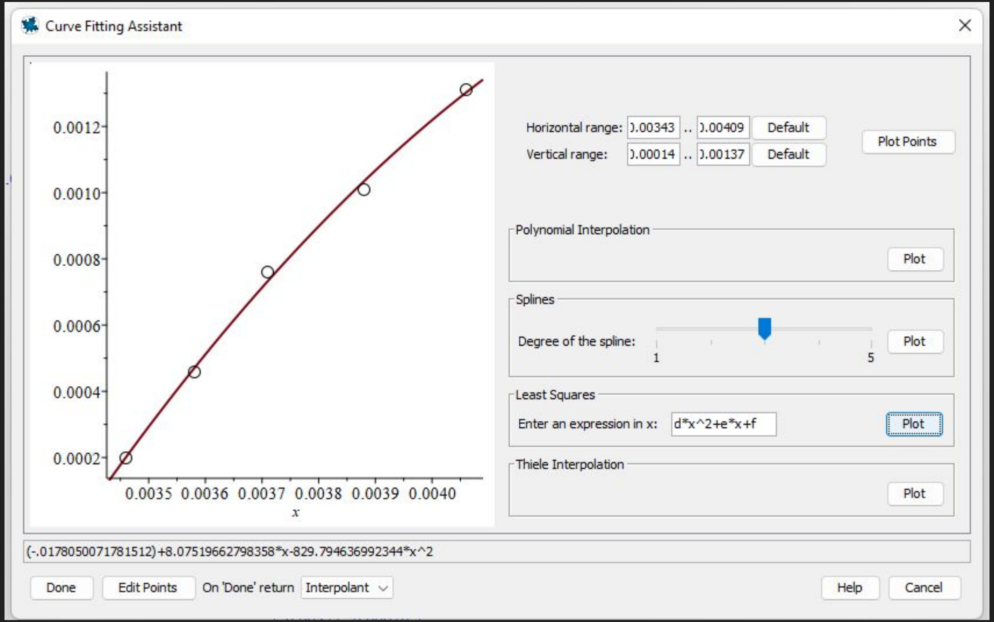
Method 1

1) Find the Right-Most and Left-Most Point from shifted coordinates
2) Separate points into top and bottom half
3) Using curve fitting, estimate a function for the top and bottom
4) Take integral of top function minus the integral of the bottom function
5) Use Conversion Factor for Longitude and Latitude

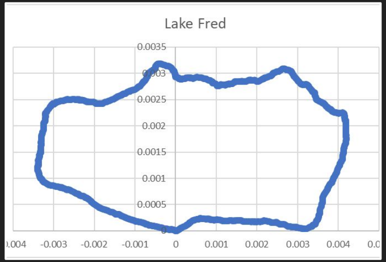


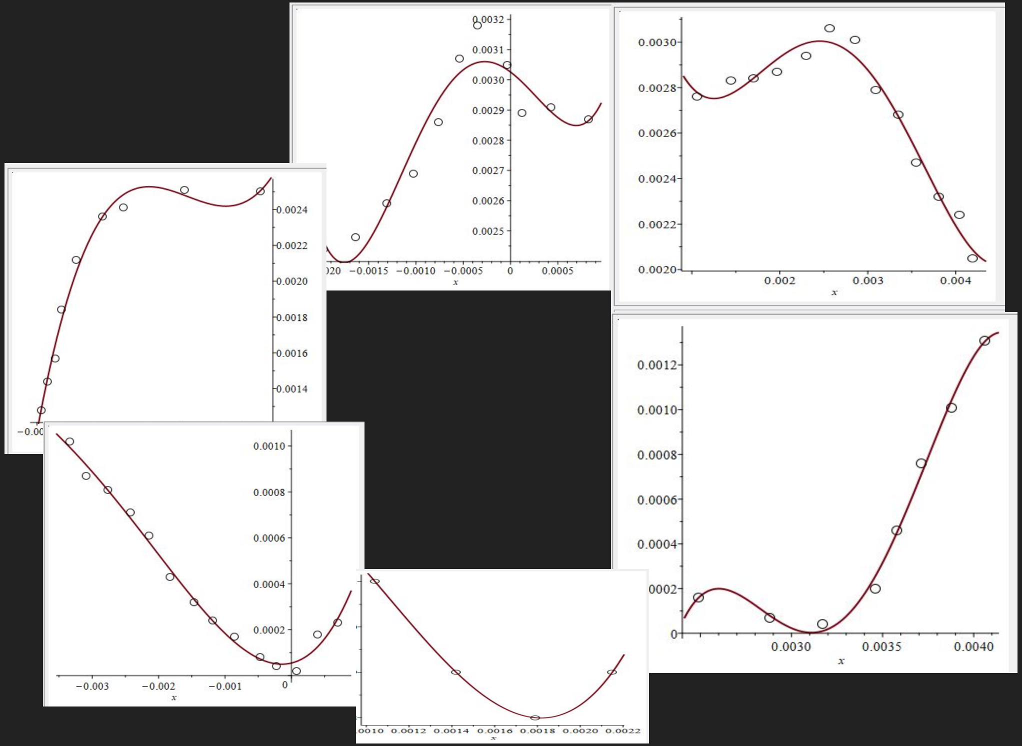
Round 1 : 6 Total Functions

- Defining each half w/ singular polynomial function was difficult
- Separated top half into 3 polynomial functions and same with bottom half
- Final calculation involved 6 integrals



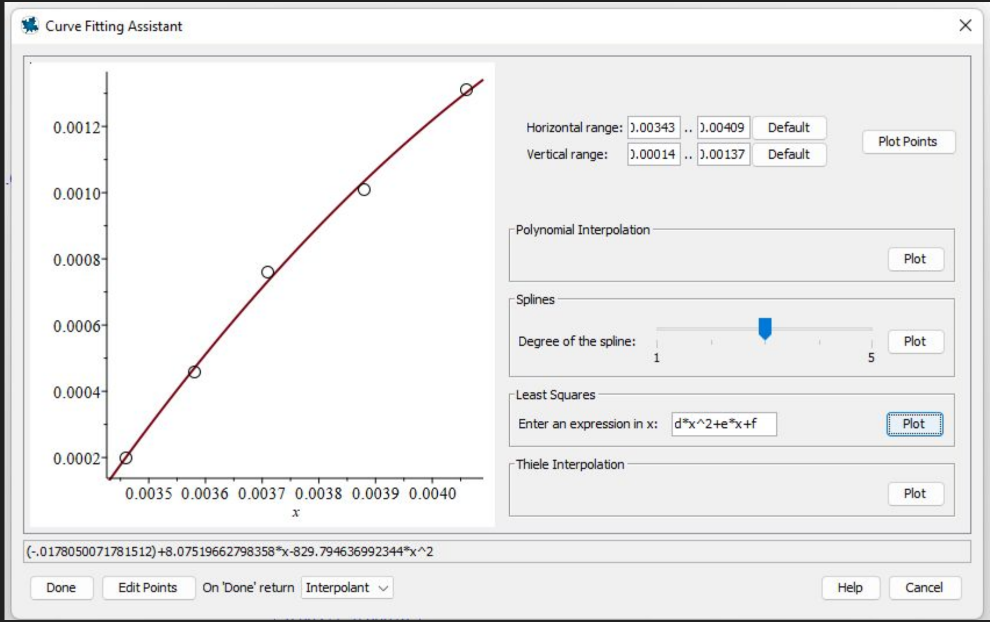
Round 1 : 6 Total Functions



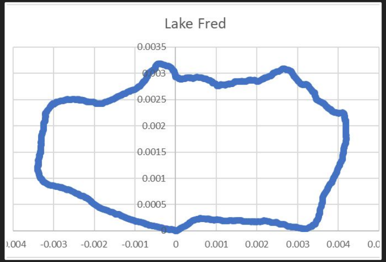


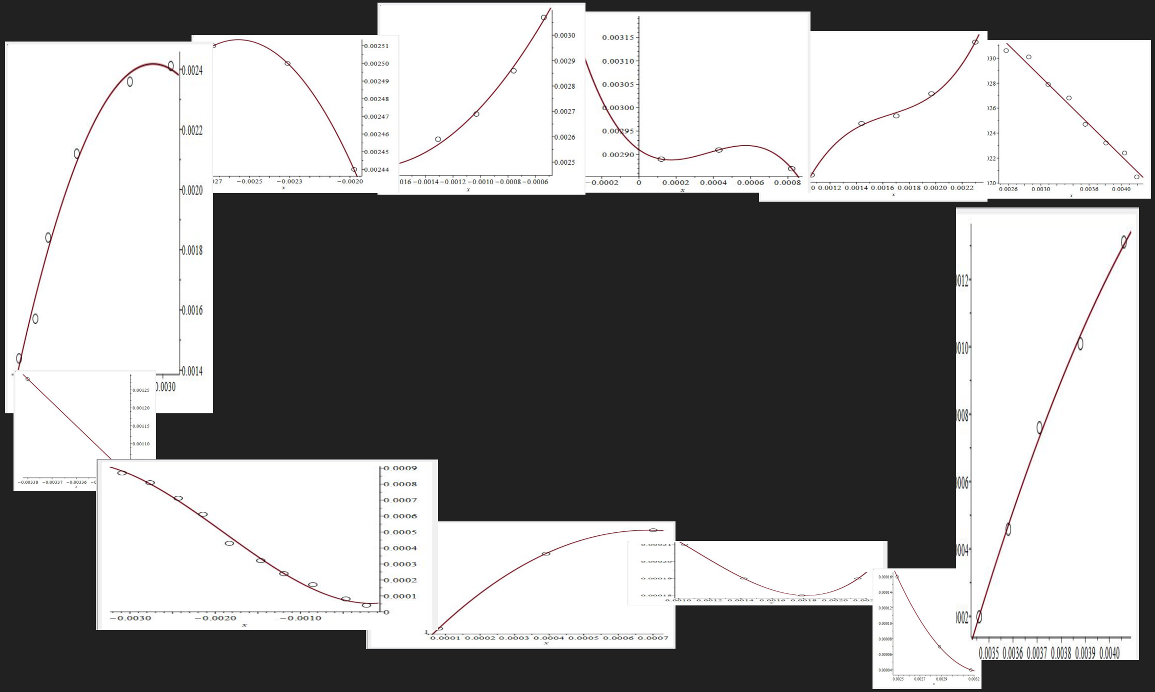
Round 2 : 12 Total Functions

- Same process as previous round now with double the functions in the top and bottom half (6 each)
- Final calculation involved 12 integrals



Round 2 : 12 Total Functions





Results

Longitude and Latitude

Round 1

- 0.000001655472217
- 203351.7859 m^2

Round 2

- 0.000001457957529
- 179089.8477 m^2

UTM

Round 1

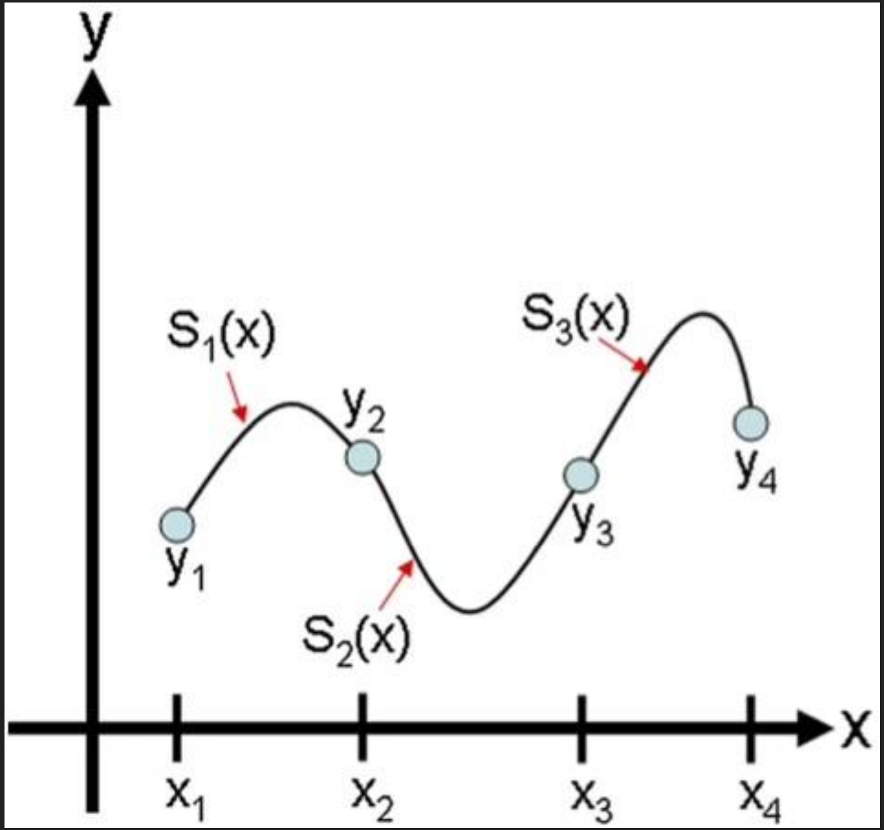
- 218205.4582 m^2

Round 2

- 219927.7535 m^2

Method 2: Cubic Spline

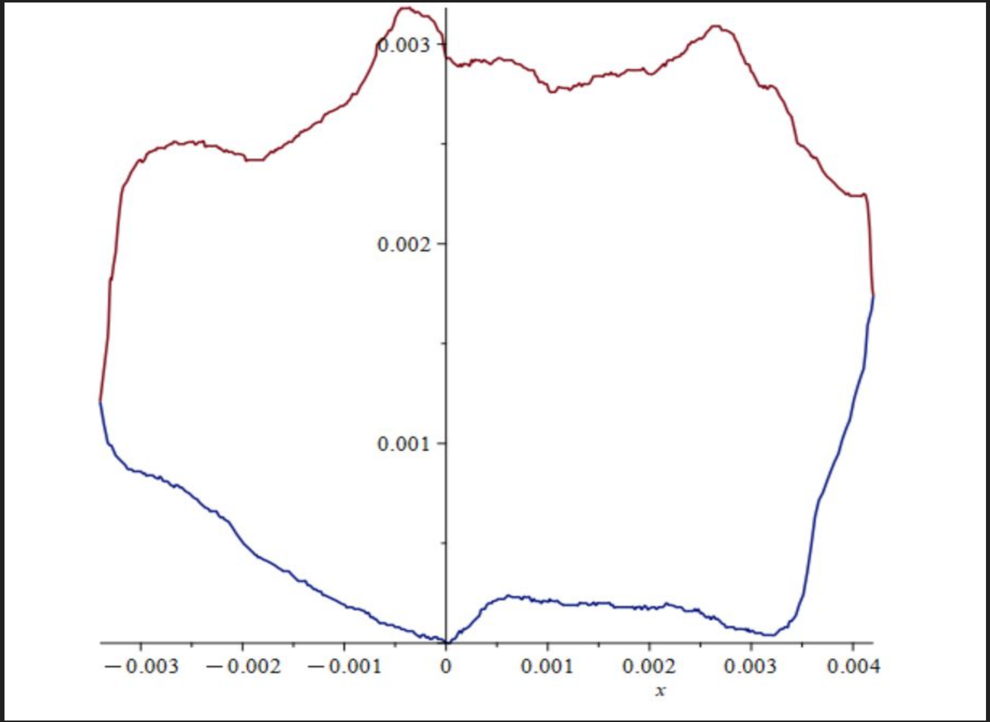
- Given some n number of points, we can approximate a function with cubic functions between each point
- Allows both f(x) and f'(x) to match at each point
- Natural Cubic Spline was taken



Results

Longitude and Latitude

- 0.00001777177028
- 218299.7280



Calculus III Method - Green’s Theorem

Green’s Theorem

- Take a simple closed curve on a plane (does not cross itself)
- The curve travels counterclockwise

Green's Theorem:

Deals w/ line integrals of simple closed curves over a non-conservative vector field.

$$\oint_C P\,dx + Q\,dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

A line integral for simple closed curve = Double integral over Region that the Curve contains

Finding Area with Green's Theorem

- so any $\oint_C = \iint_R 1 \, dA$

  gives the <u>AREA</u> of enclosed region

  $$A = \iint_R 1 \, dA \qquad \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dA$$

  $$\oint_C P dx + Q dy$$

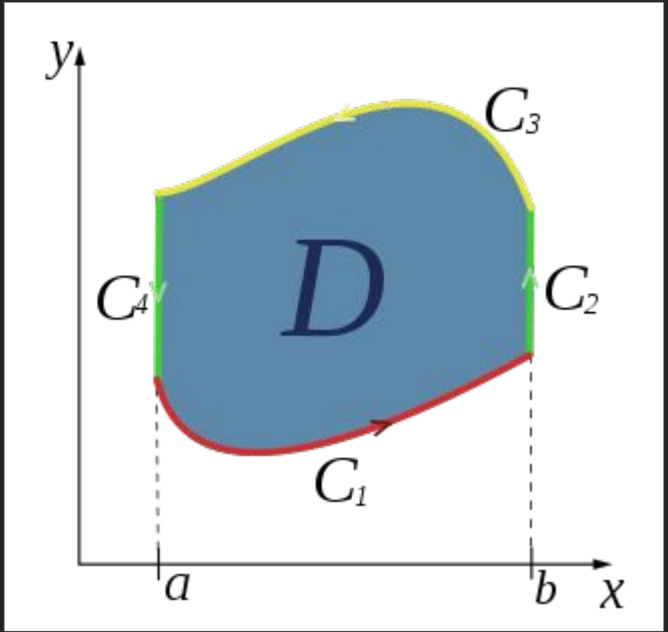
  $$= \frac{1}{2} \iint 2 \, dA$$

  $$= \frac{1}{2} \iint \Big[\underbrace{1}_{\frac{\partial Q}{\partial x}} - \underbrace{(-1)}_{\frac{\partial P}{\partial y}} \Big] dA$$

  $$A = \frac{1}{2} \oint x \, dy - y \, dx$$

General Method

1) Parameterize data as line segments:
    a) $C_1, C_2, \ldots, C_n$
2) Find t value range (integral bounds) for each C
3) Integrate using area formula
4) Use Conversion factor for Longitude and Latitude



Parameterization Method 1

- Slope-Intercept Form of a Linear Function:

  $y = mx + b$

  let $x = t$ and $y = mt + b$
  & the vector-valued function $r(t)$ be:

  $$r(t) = \langle \underset{x}{\underset{\parallel}{t}} , \underset{y}{\underset{\parallel}{mt + b}} \rangle$$

  Area would be:

  $$\frac{1}{2}\int_{t_0}^{t_1} [\, mt - (mt + b)]\,dt = -\frac{1}{2}\int_{t_0}^{t_1} b \, dt$$

  $$A = \frac{1}{2}\oint x\,dy - y\,dx$$

  For each Segment,
  where

  $t_0 = x_0$ & $t_1 = x_1$

Parameterization Method 2

- Given two points:

  (x_0, y_0) & (x_1, y_1)

  The Direction vector is:

  $$\vec{v} = \langle \underbrace{x_1 - x_0}_{a}, \underbrace{y_1 - y_0}_{b} \rangle$$

  The parameterization of a line is:

  $$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned} \quad \rightarrow \quad \begin{aligned} x &= x_0 + (x_1 - x_0)t \\ y &= y_0 + (y_1 - y_0)t \end{aligned}$$

  & vector-valued function $r(t)$ would be:

  $$r(t) = \langle x_0 + (x_1 - x_0)t, y_0 + (y_1 - y_0)t \rangle$$

Parameterization Method 2 (cont.)

- $r(t) = \langle x_0 + (x_1 - x_0)t ,\ y_0 + (y_1 - y_0)t \rangle$

  $r'(t) = \langle x_1 - x_0 ,\ y_1 - y_0 \rangle$

  Area would be:

  $$A = \frac{1}{2} \oint x\,dy - y\,dx$$

  $$\frac{1}{2} \int_{t_0}^{t_1} \left[(x_0 + (x_1 - x_0)t)(y_1 - y_0) - (y_0 + (y_1 - y_0)t)(x_1 - x_0) \right] dt$$

  $$= \frac{1}{2} \int_{t_0}^{t_1} (x_0 y_1 - y_0 x_1)\, dt$$

  For every segment, where

  $t_0 = 0$ & $t_1 = 1$

Parameterization Method 3 (Cubic Spline)

From cubic spline: $y = ax^3 + bx^2 + cx + d \quad x \in [x_0, x_1]$

Let $x = t$ and $y = at^3 + bt^2 + ct + d$, the vector-valued function $r(t)$ would be:

$$r(t) = \langle t, at^3 + bt^2 + ct + d \rangle$$

$$r'(t) = \langle 1, 3at^2 + 2bt + c \rangle = \langle dx, dy \rangle$$

Following Green's Thm: $\frac{1}{2} \oint x\,dy - y\,dx$

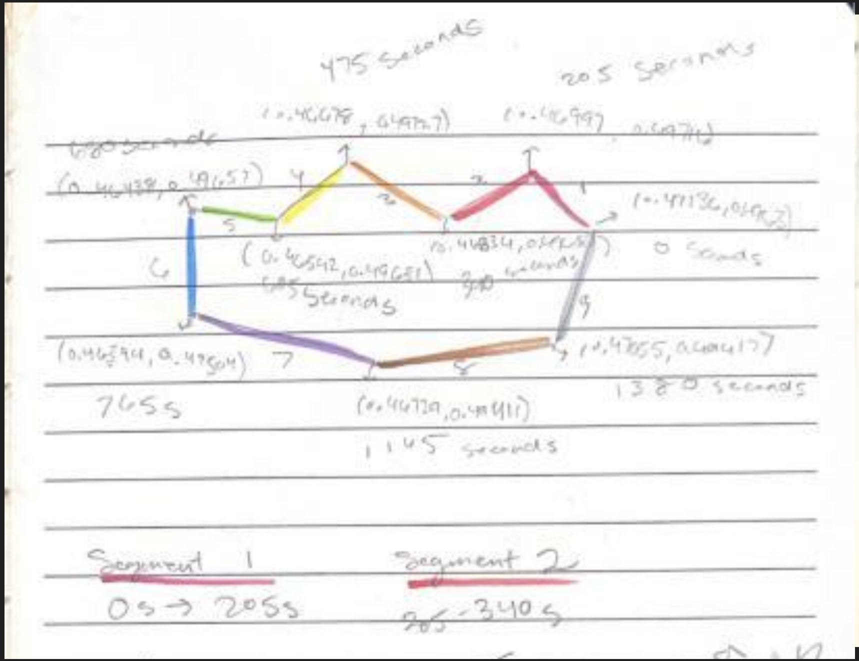
$$\rightarrow \frac{1}{2} \int_{t_0}^{t_1} 3at^3 + 2bt^2 + ct - [at^3 + bt^2 + ct + d]\,dt$$

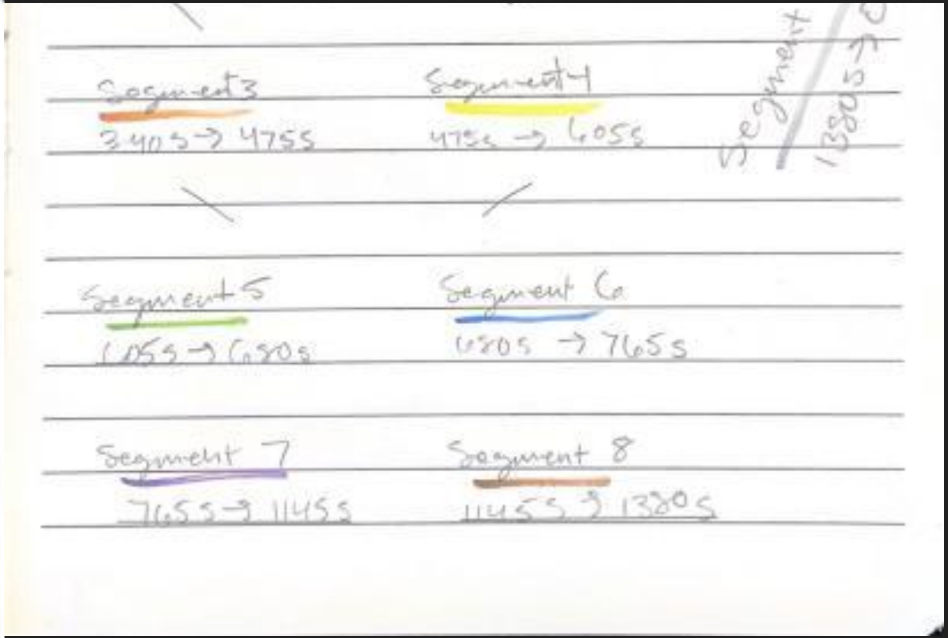
$A = \frac{1}{2} \int_{t_0}^{t_1} 2at^3 + bt^2 - d\,dt$ For each segment, where $t_0 = x_0$ & $t_1 = x_1$

Method 3 (cont.)

- Cubic spline allows the curve to be piecewise smooth
  - Can be achieved with one half of the lake being approximated with a clamped spline

Round 1 : Using 9 Line Segments





Results

Longitude and Latitude

Method 1:

- 0.00001752584927
- 215280.7345 m^2

Method 2:

- 0.000017525845
- 215280.682 m^2

UTM

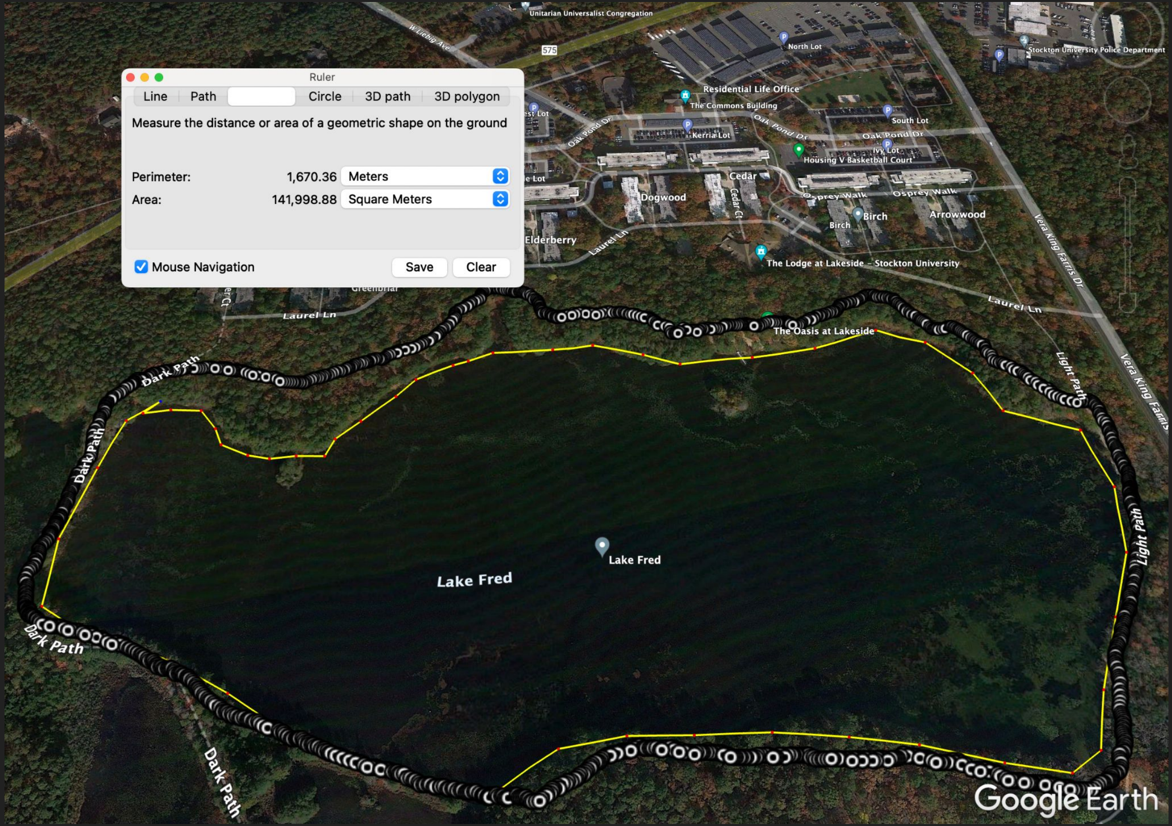
Method 1:

- 196541.500 m^2

Method 2

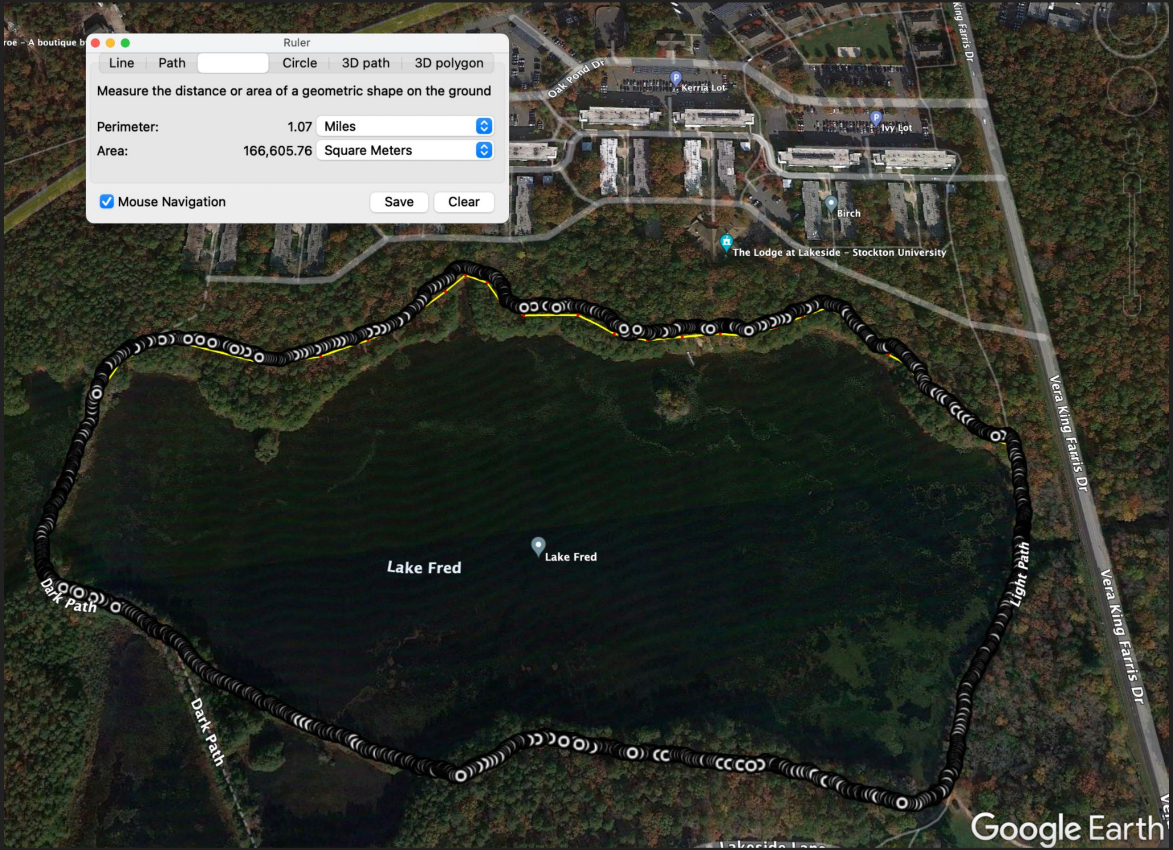
- 196541.5000 m^2

Google Earth Estimate



- $141{,}998.88\text{m}^2$ area estimate.

Google Earth Estimate With Land



- $166{,}605.76m^2$ area estimate.

Accuracy of Lake Area Measurement

Percent Error Calculation:

- Measured Area (with path): 166,605.76 m²
- Actual Area (without path): 141,998.88 m²

Formula:

$$\% \text{ Error} = \frac{|Measured Area - Actual Area|}{Actual Area} \times 100$$

Result: 17.33% Error

Results and Error: Longitude and Latitude

Calculus II

Round 1

- 0.000016554722170
- 203351.7859 m^2
- **22.0556755% Error**

Round 2

- 0.000014579575290
- 179089.8477 m^2
- **7.493190931% Error**

Calculus III

Method 1:

- 0.000017525849270
- 215280.7345 m^2
- **29.21566127% Error**

Method 2:

- 0.000017525845
- 215280.682 m^2
- **29.21562975% Error**

Results and Error: UTM

Calculus II

Round 1

- 218205.4582 m^2
- **30.97% Error**

Round 2

- 219927.7535 m^2
- **32.00% Error**

Calculus III

Method 1:

- 196541.500 m^2
- **17.97% Error**

Method 2

- 196541.5000 m^2
- **17.97% Error**

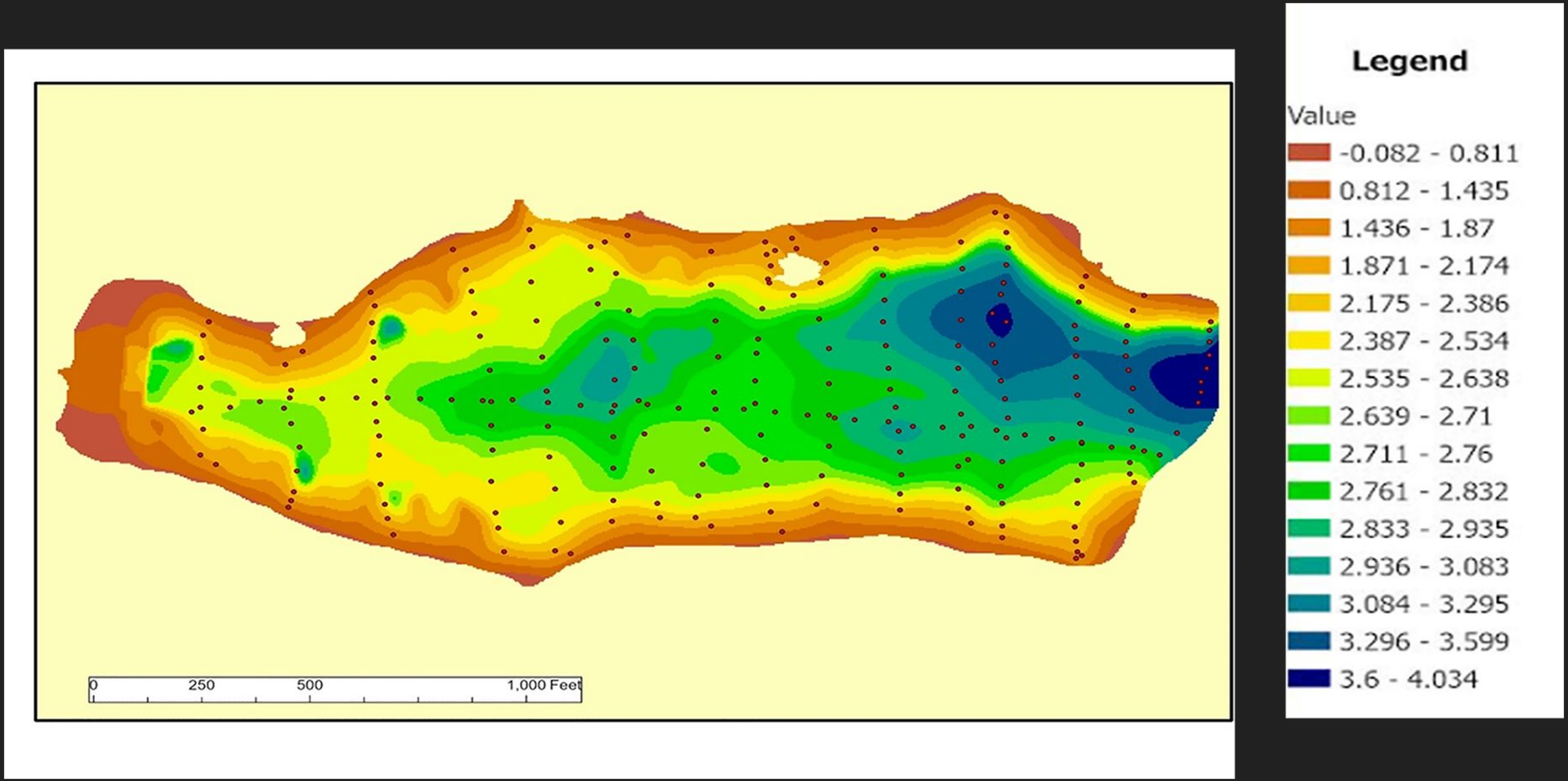
Additional Possible Sources of Error

- Calc 2 missing points between defined functions
    - Underestimate
- Round-Off Error
- Islands

Future Research Plans

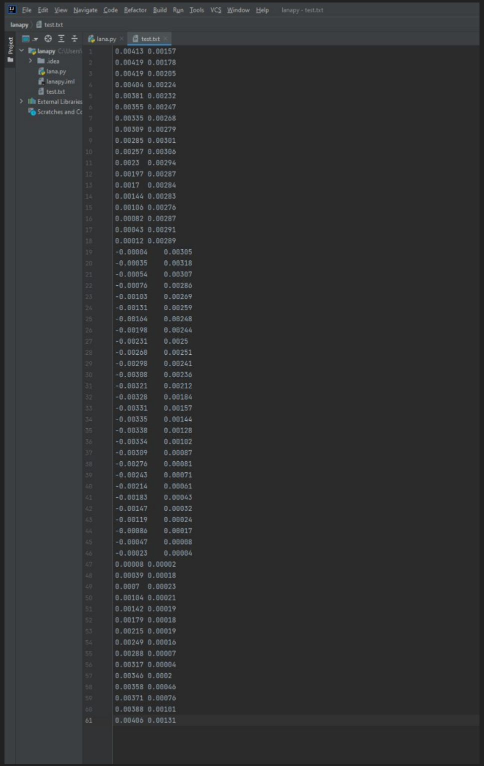
- Remeasuring Lake
  - Using LabQuest 3 device
  - Using DJI Mavic Mini drone
  - Using Google Earth
- Coding
  - Using a code to accelerate our calculations rather than by hand
- Finding Volume

Marine Science Club Collaboration - Bathymetry data

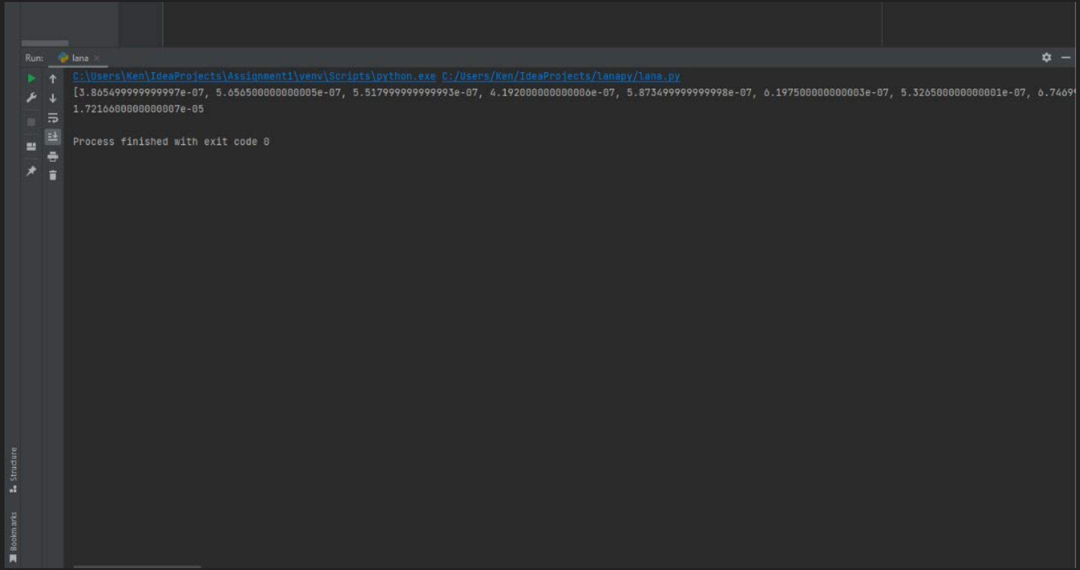


Python code – Green's Theorem

```python
from numpy import double
from scipy.integrate import quad

def integrand(x,a,b,c,d):
    return ((a-(a*x)+(b*x))*(-c+d)-(c-(c*x)+(d*x))*(-a+b))/2


data = []
dataset = []
dataUsed = []
output = 0
outputarray = []
with open('test.txt') as f:
    lines = f.readlines()
    flag = 0
    for line in lines:
        dataset.append(line.strip('\n'))
        if flag == 1:
            data.append(dataset[:])
            dataset.remove(dataset[0])
            flag = 0
        flag = flag + 1

    for x in data:
        for y in x:
            data1 = y.split(",")
            for i in data1:
                data2 = i.replace("\t"," ").split(" ")
                dataUsed.append(data2[0])
                dataUsed.append(data2[1])
        a, c, b, d = [double(x) for x in dataUsed]
        I = quad(integrand, 0, 1, args=(a,b,c,d))
        outputarray.append(I[0])
        output = output + I[0]
        dataUsed.clear()
print(outputarray)
print(output)
```



File Edit View Navigate Code Refactor Build Run Tools VCS Window Help
lanapy - test.txt
lana.py
test.txt
0.00413 0.00157
0.00419 0.00178
0.00419 0.00205
0.00404 0.00224
0.00381 0.00232
0.00355 0.00247
0.00335 0.00268
0.00309 0.00279
0.00285 0.00301
0.00257 0.00306
0.0023 0.00294
0.00197 0.00287
0.0017 0.00284
0.00144 0.00283
0.00106 0.00276
0.00082 0.00287
0.00043 0.00291
0.00012 0.00289
-0.00004 0.00305
-0.00035 0.00318
-0.00054 0.00307
-0.00076 0.00286
-0.00103 0.00269
-0.00131 0.00259
-0.00164 0.00248
-0.00198 0.00244
-0.00231 0.0025
-0.00268 0.00251
-0.00298 0.00241
-0.00308 0.00236
-0.00321 0.00212
-0.00328 0.00184
-0.00331 0.00157
-0.00335 0.00144
-0.00338 0.00128
-0.00334 0.00102
-0.00309 0.00087
-0.00276 0.00081
-0.00243 0.00071
-0.00214 0.00061
-0.00183 0.00043
-0.00147 0.00032
-0.00119 0.00024
-0.00086 0.00017
-0.00047 0.00008
-0.00023 0.00004
0.00008 0.00002
0.00039 0.00018
0.0007 0.00023
0.00104 0.00021
0.00142 0.00019
0.00179 0.00018
0.00215 0.00019
0.00249 0.00016
0.00288 0.00007
0.00317 0.00004
0.00346 0.0002
0.00358 0.00046
0.00371 0.00076
0.00388 0.00101
0.00406 0.00131



Run: lana
C:\Users\Ken\IdeaProjects\Assignment1\venv\Scripts\python.exe C:/Users/Ken/IdeaProjects/lanapy/lana.py
[3.865499999999997e-07, 5.656500000000005e-07, 5.517999999999993e-07, 4.192000000000006e-07, 5.873499999999998e-07, 6.197500000000003e-07, 5.326500000000001e-07, 6.7469
1.7216600000000007e-05
Process finished with exit code 0

Mini Discoveries!

Jefferson the Turtle